Theory and Application of Adaptive Transmission (AT\textsubscript{x}) Radar

J. R. Guerci, DARPA/SPO and S. U. Pillai, Polytech. U.
Outline

- Definition of an $AT_x$ radar
  - Potential benefits?
- Theory of optimal $AT_x$ radar
  - Maximizing SINR by jointly optimizing $T_x$ and $R_x$
  - Solution for both additive white and colored noise case
- Application to interference mitigation
  - Interference multipath problem
  - Mode of operation
- Theory of optimal $AT_x$ radar for $N$-target ID problem
  - Two target case
  - $N$-Target case
  - Application to target signature uncertainty problem
- Extension to the signal dependent noise case
  - Joint $T_x$ and $R_x$ in the presence of clutter
  - Iterative procedure for optimum transmit waveform
- Other extensions and areas for future research
Adaptive Transmitter?

- Adaptive Transmitter Definition:
  - Vary any or all of its DOFs “on-the-fly” in response to environmental stimuli
  - DOFs include:
    - Operating frequency and bandwidth (including subbanding)
    - Polarization diversity
    - Waveform, etc.
  - Assumes presence of adaptive receiver

- Potential Benefits?
  - Dynamic interference suppression?
  - Enhanced target detection?
  - Enhanced target ID?
  - Cost, complexity vs. payoff trades
Optimum $T_x/R_x$ Concept

- Choose both transmit waveform $s(t)$, and receiver $h_R(t)$ to maximize SINR
• Additive colored noise (ACN) case

- **Step 1:** Optimal receiver (in a max SINR sense), consists of a “whitening” filter followed by a matched filter (matched to the “whitened” target echo response)
  - Existence of causal and stable whitening filter guaranteed due to w.s.s. and rationality assumption

Optimum $T_x/R_x$ Concept for ACN Case

- Corresponding output SINR:

$$\text{SINR}_o = \frac{1}{\sigma_w^2} \int_{T_i}^{T_f} \left| y_w(t) \right|^2 dt = f(s(t))$$

Choose $s(t)$ to maximize $\text{SINR}_o$ (variational calculus)
Optimum $T_x/R_x$ Concept for ACN Case

- **Step 2:** Choose $s(t)$ to maximize $\text{SINR}_0$

$$\max_{s(t)} \frac{1}{\sigma_w^2} \int_{T_i}^{T_f} |y_w(t)|^2 dt$$

subject to: $s(t) \in [0, T]$, and $\int_0^T |s(t)|^2 dt < \infty$

$$\int_{T_i}^{T_f} |y_w(t)|^2 dt = \int_{T_i}^{T_f} |s(t) * h(t)|^2 dt = \int_{T_i}^{T_f} (s(t) * h(t))(s^*(t) * h^*(t))dt$$

$$= \int_{T_i}^{T_f} \left( \int_0^T s(\tau_1)h(t-\tau_1)d\tau_1 \right) \left( \int_0^T s^*(\tau_2)h^*(t-\tau_2)d\tau_2 \right)dt$$

$$= \int_0^T s(\tau_1) \int_0^T s^*(\tau_2) \int_0^{T_f} h(t-\tau_1)h^*(t-\tau_2) dt d\tau_2 d\tau_1$$

$$= \int_0^T \int_0^T K^*(\tau_1, \tau_2) d\tau_2 d\tau_1$$

where $h(t) = h_T(t) * h_w(t)$, and $K(\tau_1, \tau_2) = \int_{T_i}^{T_f} h^*(t-\tau_1)h(t-\tau_2)dt$
• **Step 2: Cont.**

\[
\max_{s(t)} \int_0^T s(\tau_1) \int_0^T s^*(\tau_2) K^*(\tau_1, \tau_2) d\tau_2 \ d\tau_1
\]

Applying Schwarz’s inequality yields:

\[
\lambda s(\tau_1) = \int_0^T s(\tau_2) K(\tau_1, \tau_2) d\tau_2
\]

• Optimal transmit waveform must satisfy a homogeneous Fredholm integral of the 2nd kind with generally Hermitian p.d. kernel (i.e., “eigensystem”)

\[
\lambda_{\text{max}} s_{\text{opt}} (t) = \int_0^T s_{\text{opt}} (\tau) K(t - \tau) d\tau
\]

**Note:** \( s_{\text{opt}} \xrightarrow{F} S_{\text{opt}}(\omega) \neq H_w^*(\omega) \) (Matched Spectrum Solution)  

(Gjessing, Farina Harger)
Multipath Interference

White Noise

Colored Noise

\[ n(t) \]

\[ h_{MP}(t) \]

\[ n_{MP}(t) = n(t) * h_{MP}(t) \]
Interference Multipath Example

\[ h_{MP}(t) = \delta(t) + 0.9\delta(t-2) + 0.5\delta(t-5) + 0.2\delta(t-10) \]

- **No Multipath**
- **Specular multipath spectrum**
- **Spectrum of optimum pulse for specular multipath case**

- **9.6 dB gain over LFM waveform and MF**
  - 8.6 dB matching gain on transmit
  - ~1.0 dB gain on receive (matched filter)
- **Relaxes channel match requirements**
Target ID Problem
**AT_x for CID**

- **AT_x concept can be applied to CID problem**

- **Choose transmit waveform to “maximally separate” echoes from different target types**

![Diagram](image)

- **Received Signal Observation Space**
- **Target-1**
  - $y_1(t)$
- **Target-2**
  - $y_2(t)$

**“Uncertainty” Sphere due to Noise, Modeling Errors, etc.**

**Distance Metric:**

$$\int |y_1(t) - y_2(t)|^2 \, dt$$
Theorem: If \( p(\mathbf{n}) \) is symmetric, unimodal and monotonic, then maximizing \( \|d\| \), minimizes the “overlap” between conditional pdfs.
Example: Two target ID problem

**OBJECTIVE**: Maximize $L_2$-norm distance between echoes:

$$\int \left| y_1(t) - y_2(t) \right|^2 dt$$

- Transmit Waveform
- $s(t)$
- $h_1(t)$ Target-1
- $h_2(t)$ Target-2
- $y_1(t)$
- $y_2(t)$
- $y_1(t) - y_2(t)$
- $\sum$
- $t = T_f$

Equivalent to single system with impulse response: $h(t) = h_1(t) - h_2(t)$

**Optimum $s(t)$**

$$\lambda s(t) = \int_0^T s(\tau) K(t-\tau) d\tau$$

where:

$$K(t-\tau) = \int_{\tau_i}^{T_f} h^*(t-\alpha) h(\tau-\alpha) d\alpha$$
Maximize Average Separation:

\[
\lambda s(t) = \int_0^T s(\tau) K(t - \tau) d\tau
\]

\[
\bar{K}(t - \tau) = \sum_{m,n}^{N} \int_{T_i}^{T_i^*} h_{T_{m,n}}^*(\alpha - t) h_{T_{m,n}}(\alpha - t) d\alpha
\]

\[
h_{T_{m,n}}(t) = h_{T_m}(t) - h_{T_n}(t)
\]

Weighting can be introduced to ascribe relative importance

Maximize "Volume" Spanned
Optimal Receiver Structure

- \( N \)-Target case (AGWN)

\[
\begin{align*}
\text{Received Signal} & \rightarrow h_{R_1}(t) \quad \text{Target-1} \\
& \rightarrow h_{R_2}(t) \quad \text{Target-2} \\
& \quad \vdots \\
& \rightarrow h_{R_N}(t) \quad \text{Target-}N \\
& \rightarrow g_1(t) \\
& \rightarrow g_2(t) \\
& \rightarrow g_N(t) \\
& \rightarrow \text{Likelihood Comparator} \quad \text{Selection}
\end{align*}
\]

- Replace MF with WMF for AGCN case
Example 1: Three “Targets”

\[ h_1(t) \quad h_2(t) \quad h_3(t) \]

\[ \text{Chirp: } P_{cc} = 58.40\% \]
\[ \text{SNR} = 0 \text{ dB} \]
\[ \text{Optimum: } P_{cc} = 99.99\% \]

1000 Monte Carlo Trials
$\text{AT}_x$ for Target Tracking/Tagging

M1

T72

M1 HRR Profile

T72 HRR Profile

Front Aspect

* Data provided by D. Gerren and D. Andersh, SAIC
**AT\_x for Target Tracking/Tagging**

\[
\frac{d_{\text{opt}}}{d_{\text{chirp}}} = 3.98 \text{ (dB)}, \quad \frac{d_{\text{matched}}}{d_{\text{chirp}}} = 1.72 \text{ (dB)}
\]
Target Aspect Uncertainty

- Case I: Target Known to be Present

Maximize Average Separation:

\[ \lambda_s(t) = \int_0^T s(\tau) \overline{K}(t - \tau) d\tau \]

\[ \overline{K}(t - \tau) = \sum_{m,n} \int_{T_i}^{T_f} h_{T_{m,n}}(\alpha - t) h_{T_{m,n}}^*(\alpha - t) d\alpha \]

\[ h_{T_{m,n}}(t) = h_{T_m}(t) - h_{T_n}(t) \]
• Case II: Composite Detection and ID

**Target Aspect Uncertainty**

- Image of a diagram illustrating received signal observation space with aspects 1, 2, and N. The null hypothesis is shown connecting the aspects.

- Equations:
  
  \[ y_1(t) \rightarrow r(t_1) \]
  
  \[ y_2(t) \rightarrow r(t_2) \]
  
  \[ y_N(t) \rightarrow r(t_N) \]
  
  \[ y(t) \rightarrow r(t) \]
• Problem
  – Maximum “eigenfunction pulse” may have undesirable properties
    • Poor resolution
    • Poor range-sidelobes
    • Complicated modulation (transmitter inefficiency)
  – Including constraints directly into objective function can lead to highly nonlinear and inhomogeneous integral equations
    • Difficult to solve (even numerically)
• “A” Solution
  – Use “principal” eigenfunctions of target (autocorrelation) kernel to synthesize a more suitable pulse
    • Eigenfunctions of kernel form a complete $L_2$-norm basis
    • Project desired (unmatched, e.g., LFM) pulse onto PC’s
    • Gradually trade-off matching gain for constraints
Constrained $AT_x$

Eigenfunctions of:
\[
\lambda_i \phi_i(t) = \int_0^T \phi_i(\tau) K(t-\tau) d\tau
\]
where
\[
K(t-\tau) = \int_{T_i}^{T_f} h_T^*(t-\alpha) h_T(\tau-\alpha) d\alpha
\]

Form a complete (in general) orthonormal basis in $L_2$:
For any $f(t) : \int_0^T \left| f(t) \right|^2 dt < \infty$
\[
f(t) = \sum_{i=1}^M \langle \varphi_i(t) \mid f(t) \rangle \varphi_i(t)
\]
where \[
\langle \varphi_i(t) \mid f(t) \rangle = \int_0^T \varphi_i^*(t) f(t) dt
\]
A pulse compression modified OMIR waveform $f(t)$ is obtained by computing the OMIR eigenfunctions $\psi_i$, $i=1, 2, \ldots, \infty$, for an autocorrelation function of the expected target impulsive response, specifying a waveform $c(t)$ having a desired pulse compression characteristic, and generating expansion terms

$$f(t) = \sum_{i=1}^{\infty} a_i \psi_i(t)$$

for various expansion indices $N$, until a desired waveform is obtained. The expansion coefficients $a_i$ are given by

$$a_i = \int_0^T f(t) \psi_i(t) dt$$

Constrained Waveform w/ 10 Components

1.0 dB Gain Over Chirp

Unconstrained Optimum Waveform

1.5 dB Gain Over Chirp
Signal-Dependent Noise (Clutter)

- Choose both transmit waveform $s(t)$, and receiver $h_r(t)$ to maximize SINR
Signal-Dependent Noise (Clutter)

\[ \text{SINR}_o = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} H_R(\omega) H_T(\omega) S(\omega) e^{-j\omega T_f} d\omega \right|^2 \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H_R(\omega) \right|^2 \left( G_n(\omega) + G_c(\omega) \left| S(\omega) \right|^2 \right) d\omega \]

**Case I:** \( G_c(\omega) << G_n(\omega) \) (Previous solution)

**Case II:** \( G_c(\omega) >> G_n(\omega) \) (Only requirement on transmit is minimum phase!)
- Still requires fully matched \( R_x \) (Urkowitz filter results)
- Free to maximize time-bandwidth resolution (Manasse)

**Case III:** \( G_c(\omega) \sim G_n(\omega) \) (Effective iterative procedure developed)

• Case III: $G_c(\omega) \sim G_n(\omega)$ (Iterative Procedure)

\[
S_0(t) \quad \text{Satisfies energy and duration constraints}
\]

Wiener-Hopf Factor
\[
|L(j\omega)|^2 = G_n(\omega) + G_c(\omega)S_k(\omega)^2
\]

Derive Eigensystem
\[
h_k(t) \leftrightarrow L_k^{-1}(j\omega)H_T(\omega)
\]
\[
K(\tau_1, \tau_2) = \int h_k^*(t - \tau_1)h_k(t - \tau_2)dt
\]

Solve Eigensystem
\[
\lambda_1^{(k)}\psi_1^{(k)}(\tau_1) = \int K(\tau_1, \tau_2)\psi_1^{(k)}(\tau_2)d\tau_2
\]

Update
\[
s_{k+1}(t) = \kappa_k(s_k(t) + \varepsilon_k\psi_1^{(k)}(t))
\]
Signal-Dependent Noise (Clutter)

Target Response

Clutter and Noise psd

Convergence Rate

SINR vs. Iteration

Starting Pulse

First

Final Pulse

Final Matched Filter
Summary and Future Research

- **Mathematical framework for optimum \( \text{AT}_x \)**
  - Max SINR based on target and interference
  - Interference multipath example
  - Application to CID and target signature uncertainty
  - Constrained solution based on eigendecomposition
  - Signal-dependent noise case (clutter)

- **Future research**
  - Constrained optimization (constant modulus)
  - Examine potential in other realistic applications
  - Multichannel applications (especially polarization)
    - Multichannel theory to be presented at *SAM 2000*
  - Bi/Multi-static configurations
Adaptive Radar Timeline

CACFAR  AGC, etc.
IF Sidelobe Canceler
Fully Adaptive Array
Space-Time Adaptive Radar
Advanced STAP and Real-Time Formulations

50’s  60’s  70’s  80’s  90’s  00’s

Adaptivity in the Receiver

Adaptivity in the Transmitter?