Root Locus Properties of Adaptive Beam Forming and Capon Estimation for Uniform Linear Arrays

A. Steinhardt / Alphatech
L. Scharf (as of 8:24am,3/17)/CSU
Problem and result

Let $\vec{v}$ be a length N Vandermonde steering vector,

$$\vec{v}_{\omega_t} = [1, \exp(j \omega_t), \ldots, \exp(j \omega_t (N - 1))]^T$$

where $\omega_t$ is the target arrival angle in normalized coordinates.

It is well known that this vector has exactly N-1 nulls, i.e., its Z transform has all unit circle roots:

$$V(z) = \sum_{i=1}^{N} \text{Exp}(j \omega i)z^{-i} = \frac{1 - (\text{Exp}(j \omega)z^{-1})^{N-1}}{1 - (\text{Exp}(j \omega)z^{-1})}$$

Let $R$ be a Toeplitz matrix (sample matrix for interference), and let us form the SMI MVDR weight vector:

$$\vec{w} = \frac{R^{-1}\vec{v}}{\vec{v}^H R^{-1} \vec{v}}$$

Theorem: the weight vector $w$ has all its roots on the unit circle.
Matlab examples

Matlab “proves” the theorem. Matlab shows that nulls drift if array isn’t linear: multipath or manifold error predictor.
Why we care

Even mainbeam nulling never leads to finite nulls!
Proof

• Lemma: This is a surprising result!
  – Proof of Lemma: Dr Guerci and Dr Zatman think so!
  – ALL NULLS ALWAYS INFINITELY DEEP!!!!!!

• Proof of theorem: MVDR solves

\[
\min_{\tilde{w}} \tilde{w}^H R \tilde{w} \equiv f
\]

Wiener Khintchine, objective

\[
f = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) \left| \sum_{i=1}^{N} w_i e^{-j\omega i} \right|^2 d\omega ~ S(\omega) > 0 \forall \omega
\]

Or

\[
f = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) W(z)W^*(z^{-1}) d\omega, Z = \exp(j\omega) \quad \text{with} \quad W(1) = 1
\]

Let J be anti-identity. The JRJ=R, JRJw=v=Jv, so w=Jw

Hence roots appear as reciprocals. Are they unit modulus?

\[
W(Z) = \prod_{i=1}^{n} \frac{(1 - z_i z^{-1})}{(1 - z_i)}, W(1) = 1
\]
Reformulation of MVDR cost function:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) \left| \prod_{i=1}^{n} \frac{(1 - z_i z_i^{-1})}{(1 - z_i)} \right|^2 d\omega
\]

If I replace a root by its inverse, constraint is preserved, and if root is NOT on the unit circle I have a different weight vector. But weight vector is unique by convexity. Hence we invoke reductio ad absurdum.

QED