MAXIMUM LIKELIHOOD ESTIMATION OF POINT SCATTERERS FOR
COMPUTATIONAL TIME-REVERSAL IMAGING

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ABSTRACT

The time reversal approach, which traces its origin to “phase conjugation” in nonlinear optics, was later developed using acoustic experiments and now attracts increasing interests in diverse fields. We present a statistical framework for the fixed-frequency computational time-reversal imaging problem assuming point scatterers in a known background medium. We develop maximum likelihood (ML) estimators of the locations and reflection parameters of the scatterers. Using a simplified single-scatterer model, we also propose a likelihood time-reversal imaging technique which is suboptimal but computationally efficient and can be used to initialize the ML estimation. We generalize the fixed-frequency likelihood imaging to multiple frequencies, and demonstrate its effectiveness in resolving the grating lobes of a sparse array. This enables to achieve high resolution by deploying a large-aperture array consisting of a small number of antennas while avoiding spatial ambiguity. Numerical examples illustrate the key results.

1. INTRODUCTION

The time reversal approach, which traces its origin to “phase conjugation” in nonlinear optics [1], was later developed using acoustic experiments [2] and now attracts increasing interests with broad applications. The key idea behind the so-called physical time-reversal methods is to record a signal emitted by sources or reflected by targets using an array of transducers; then transmit the time-reversed and complex conjugated version of the measurements back into the medium. In a reciprocal medium, the back-propagated wave will then retrace the original trajectory and focus around the original source locations without the need to solve the inverse of the channel. One of many applications of this time-reversal refocusing property is to detect and locate a target by computational or virtual time reversal through imaging [3], [4]. In this case, after receiving the signal reflected from the target, a back-propagated process is computed rather than implemented in the real medium. The reversed signal then refocus in the computed image around the target location, and the peaks indicate existence of possible targets. Various computational imaging strategies have been proposed, which in general fall under three categories: (a) time domain methods that use mostly times of arrival and amplitude information recorded by the array, (b) fixed-frequency methods that use mostly differential phase information on the array, and (c) intensity measurement based methods (see [3] for more details).

In Section 2, we employ two physical models used in computational time-reversal imaging methods assuming point scatterers: the distorted wave Born approximation [5] and the Foldy-Lax models [6], [7]. We then formulate the computational time-reversal imaging in a statistical framework by establishing a measurement model. In Section 3 we present our maximum likelihood (ML) and suboptimal estimators. In Section 4 we propose a likelihood computational time-reversal imaging method using a simplified single-scatterer model, and generalize it into multiple-frequency version in Section 5, which is demonstrated to be useful in resolving the spatial ambiguity of a sparse array. Numerical examples are presented in Section 6, and conclusions are given in Section 7. For more details see [8].

2. PHYSICAL AND STATISTICAL MEASUREMENT MODELS

We present physical and statistical measurement models of the multistatic response matrix [9] made by an array of antennas, which we will employ in later sections for solving the computational time-reversal imaging in a statistical signal processing framework. We consider transmit and receive antenna arrays of \( N_t \) and \( N_r \) isotropic point antennas, centered at known positions \( \alpha_1, \alpha_2, \ldots, \alpha_{N_t} \) and \( \beta_1, \beta_2, \ldots, \beta_{N_r} \), respectively. There are \( M \) point scatterers located at \( x_1, x_2, \ldots, x_M \) with scattering coefficients \( \tau_1(\omega), \tau_2(\omega), \ldots, \tau_M(\omega) \).

2.1. Multistatic Matrix Using the Distorted Wave Born Approximation

We employ the distorted wave Born approximation (DWBA) [5], meaning that we neglect the multiple scattering among the scatterers. Suppose a known signal \( s(t) = [s_1(t), s_2(t), \ldots, s_{N_t}(t)]^T \), whose Fourier transform is \( s(\omega) = [s_1(\omega), s_2(\omega), \ldots, s_{N_t}(\omega)]^T \), is transmitted to illuminate the scenario of interest, i.e., the \( j \)-th antenna transmits \( s_j(t) \), \( j = 1, \ldots, N_t \), and the resulting backscattered returns are measured by all the receive antennas. Then, the scattered field at a position \( \mathbf{r} \) induced by the \( k \)-th transmit antenna...
where $k$ is given in frequency domain by \[ (10) \]

$$K_{j,k} = \sum_{m=1}^{M} G(\beta_j, x_m) \tau_m G(x_m, \alpha_k),$$

where $j = 1, 2, \ldots, N_j$ and $G(\cdot)$ is the scalar time harmonic background Green function [3]. The so-called multistatic matrix [9] $K(\omega)$ is then found as

$$K_{j,k}(\omega) = \sum_{m=1}^{M} G(\beta_j, x_m, \omega) \tau_m G(x_m, \alpha_k, \omega) s_k(\omega),$$

where $j = 1, 2, \ldots, N_j$ and $k = 1, 2, \ldots, N_k$. Define $x = [x_1^T, x_2^T, \ldots, x_M^T]^T$ as the unknown scatterer location parameter vector of dimension $3M$, and $\tau(\omega) = [\tau_1(\omega), \tau_2(\omega), \ldots, \tau_M(\omega)]^T$ unknown scattering coefficients. The above model is formulated in a matrix form as

$$K(x, \tau, \omega) = \sum_{m=1}^{M} \tau_m G(x_m, \alpha_k, \omega) s_k(\omega),$$

where $\tau(\omega) = [\tau_1(\omega), \tau_2(\omega), \ldots, \tau_M(\omega)]^T$ unknown scattering coefficients.

Solving $A(x) = A_1(x) + S(x)T(\tau)$ from (12) and substituting it into (10), we derive the closed-form matrix $K$ that includes the multiple scatterings as

$$K(x, \tau) = A_1(x) + S(x)T(\tau)A_1^T(x).$$

### 2.3. Statistical Measurement Model

In the fixed-frequency computational time-reversal imaging, the multistatic matrix is evaluated at only one specific frequency. Obviously the measurement and modeling of the multistatic matrix will have inaccuracies, hence we assume that it is perturbed by an additive noise. The statistical measurement model of the multistatic matrix is

$$Y = K(x, \tau) + W,$$

where $Y$ is the $N_t \times N_t$ measurement matrix, $W$ is a noise matrix whose entries $w_{ij,k}$ are assumed zero-mean jointly circularly symmetric complex Gaussian distributed, and are independent identically distributed.

Note that the Green function representations in the multistatic matrix $K(x, \tau)$ are completely general in the sense that the formulation could easily be adapted to modeling different operating scenarios by applying appropriate Green functions. Furthermore, since we make no assumptions on the antenna locations,
our model can include various calibrated array configurations for instance, linear, planar, three-dimensional, etc., as long as the coherence among the antennas are preserved. Throughout this paper, we assume the number of scatterers $M$ is a priori known, otherwise it could be estimated by examining the profile of the singular values of the multistatic matrix [12], or determined according to the information theoretic criteria [13], [14].

3. SCATTERING PARAMETER ESTIMATION

Using the above statistical measurement models for the fixed-frequency computational time-reversal imaging methods, we develop maximum likelihood (ML) and suboptimal methods for estimating the location $x$ and scattering coefficients $\tau$ vectors. Given the measurement $Y$, the MLE of $x$, $\tau$ coincide the ordinary least squares solution

$$\hat{x}, \hat{\tau} = \arg\min_{x, \tau} \|Y - K(x, \tau)\|^2_F. \quad (15)$$

To find the MLEs of $x$ and $\tau$ for the three-dimensional problem, we would need to solve the $5M$-dimensional nonlinear least-squares optimization problem in (15).

3.1. Estimation Using the Distorted Wave Born Approximation

When the multiple scatterings among the scatterers can be assumed to be weak and negligible, we apply the physical model of the multistatic matrix using the distorted wave Born approximation. Substituting (4) into the cost function in (15), we have the following metric for estimating locations $x$ and scattering coefficients $\tau$

$$l_1(x, \tau; Y) = \|Y - A_{\Phi}(x)T(\tau)A_0^T(x)\|^2_F = \|\text{vec}(Y) - A_{\Phi}(x)\otimes A_0(x)\|^2_F, \quad (16)$$

where $\text{vec}(\cdot)$ stacks the first to the last columns of the matrix one under another to form a long vector, $A_{\Phi}(x) \otimes A_0(x) = [g_1(x) \otimes g_1(x_1) \cdots g_1(x_{n_1}) \otimes g_1(x_{n_1})]$, here $\otimes$ stands for the Khatari-Rao product, and $\oplus$ represents the Kronecker product [15]. In the second equality, we used the identity $\text{vec}(AVD) = (D^T \otimes A)\text{vec}(V)$ where $V$ is diagonal and $\text{vec}(\cdot)$ forms a vector from the diagonal elements of the matrix (see T3.13 in [15]).

Observe the linearity of the vector in (17) in $\tau$. Hence, given $x$ we estimate $\tau$ as a function of $x$ and $Y$ using the ordinary least-squares solution. We further concentrate (17) with respect to the location parameters $x$

$$l_2(x; Y) = \|\text{vec}(Y) - A_{\Phi}(x)\|_{F}^2 = \|P_{A_{\Phi}(x)}\text{vec}(Y)\|^2_F, \quad (18)$$

where $P_{A_{\Phi}(x)} = I - A_{\Phi}(x)[A_{\Phi}^H(x)A_{\Phi}(x)]^{-1}A_{\Phi}^H(x)$ is the projection matrix that projects to the orthogonal complement subspace of the range of matrix $A_{\Phi}(x) \oplus A_0(x)$. Now the dimension of the optimization problem has been reduced from $5M$ in (17) to $3M$ in (18).

Observing (18), the likelihood-based optimization problem using the physical model (4) and measurement model (14) could be eventually interpreted as a subspace-based method: the vector $\text{vec}(Y)$ representing the measured signal subspace should be orthogonal to the noise subspace $P_{A_{\Phi}(x)}$, the orthogonal complement subspace of the range of $A_{\Phi}(x) = A_0(x) \otimes A_0(x)$ in $\mathbb{C}^{N_0N_1}$.

3.2. Estimation Using the Foldy-Lax Multiple Scattering Model

For the case where multiple scattering among the scatterers are non-negligible, for example when the scatterers are closely located and their scattering amplitudes are large, it is necessary to apply an appropriate physical model incorporating the underlying multiple scattering effect to obtain accurate estimation. We will then use the physical model (13), which is based on the Foldy-Lax multiple scattering model. Since $K(x, \tau)$ is nonlinear with respect to both the location parameters $x$ and scattering coefficient parameters $\tau$, it is difficult to reduce the dimension of the optimization problem (15) by finding a concentrated metric as we did in (18), and it is inevitable to resort to an iterative algorithm or gradient-based numerical procedure.

Here, we propose a sub-optimal estimation method, which is a tradeoff between computation complexity and optimality. It is interesting to note that when using the wave Born approximation, the signal subspace for the measurement vec$(Y)$ is the range of $A_{\Phi}(x) = A_1(x) \otimes A_1(x)$, which is spanned by $g_1(x_{m}) \otimes g_1(x_{m}), m = 1, 2, \ldots, M$; whereas, when multiple scattering is not negligible, the signal subspace becomes the range of $A_\Phi(x) \oplus A_1(x) \otimes A_1(x)$, which is spanned by $g_1(x_{m}) \otimes g_1(x_{m}), m, m' = 1, 2, \ldots, M$. By this observation, we propose a sub-optimal estimator of $x$ for the Foldy-Lax model that minimizes the following metric

$$l_3(x; Y) = \|P_{A_\Phi(x)} \text{vec}(Y)\|^2_F = \|(P_{A_\Phi(x)} \otimes P_{A_1(x)}) \text{vec}(Y)\|^2_F. \quad (19)$$

The metric (19) is essentially a modification of (18) by applying appropriate subspace for the multiple scattering case, but the dimension of the signal subspace increases from $M$ to $M^2$.

4. LIKELIHOOD TIME-REVERSAL IMAGING

In this section, we propose an imaging metric for the fixed-frequency time-reversal imaging methods using the likelihood function for a simplified physical model of the multistatic matrix, namely for a single-scatterer model where

$$K(x_1, \tau_1) = \tau_1 g_1(x_1)g_1^H(x_1). \quad (20)$$

That is, we apply this model even if in reality the number of scatterers is larger than one, thus reducing the optimization to a dimension of five. The resulting simplified estimation or scanning scheme is similar to [16] and is useful for example to initialize the numerical optimization of theMLE for the full physical model.

Note that this single-scatterer model is a special case of the one using the distorted wave Born approximation (4) (of course, also a special case of the one using Foldy-Lax model (13)) and location parameter $x = x_1$. We plug $A_1(x) = g_1(x)$ and $A_0(x) = g_1(x)$ into (19), the concentrated cost function for solving $x$ is

$$l_4(x; Y) = \|P_{g_1(x)} \text{vec}(Y)\|^2_F \quad (21)$$

$$= \|(I - g_1(x)g_1^H(x) \otimes g_1(x)g_1^H(x)) \text{vec}(Y)\|^2_F$$

$$= \|Y - g_1(x)g_1^H(x)Yg_1(x)g_1^H(x)/g_1^H(x)g_1(x)\|= \|g_1^H(x)g_1(x)\|^2_F, \quad (22)$$
where the last equality follows by the identities $\text{vec}(ABD) = (B^T \otimes A) \text{vec}D$, $\text{vec}(A + B) = \text{vec}(A) + \text{vec}(B)$, and $\|A\|_F = \|\text{vec}(A)\|_F$.

Now, the dimension of the optimization problem in (22) has been reduced to 3, rather than $3M$ searches necessary in (18) and (19). We propose to create the likelihood time-reversal image by evaluating the following metric over a fine grid of the probed scenario

$$l_s(x; Y) = \frac{1}{l_4(x; Y)} = \frac{1}{\|P_{l_t}^g(x)g_t(x)\|_F^2},$$

(23)

and use it as the imaging metric. Since this imaging is based on the simplified physical model (20), it is sub-optimal in its nature, but it useful also to initialize the non-linear optimization procedures in Section 3.1 and Section 3.2 by using the locations of the first $M$ local maxima as the initial estimates of $x_1, x_2, \ldots, x_M$.

We note that in [10], the following MUSIC-based pseudo-spectrum is used as a fixed-frequency computational time-reversal imaging metric for the co-located transmit and receive arrays case, (where $N_t = N_r = N$, and $g_t(x) = g_r(x) = g(x)$)

$$D(x) = \frac{1}{\sum_{m=0}^{N-1} |(\mu_{m_0}, g(x))|^2},$$

(24)

where $\mu_{m_0}$ is the $m_0$-th eigenvector of time-reversal matrix $T = Y^*Y$ having zero eigenvalue. This MUSIC algorithm makes use of the fact that the time-reversal matrix $T$ has the same range as the subspace spanned by the complex conjugates of the Green function vectors when noiseless, which is the so-called signal subspace in [10], and the noise subspace is spanned by the eigenvectors of $T$ corresponding to zero eigenvalues. Here, the orthogonality between the signal subspace and noise subspaces employed in (18) and (21) is, however, different the MUSIC-based (24). In the likelihood-based (21), both the signal and noise subspaces are in $C^{N_tN_r}$, the noise subspace represented by $P_{l_t}^g(x)g_t(x)$ arises from the physical model (20) and the signal subspace from the measurement $\text{vec}(Y)$, whereas in the MUSIC-based (24), the signal and noise subspaces are in $C^{N_t}$, the noise subspace is estimated from the measurement $Y$ and the signal space comes from the Green function vector $g(x)$. In addition, during the likelihood-based imaging process the noise subspace is a function $x$, thus evaluated at every imaging position, but in the MUSIC-based imaging the signal subspace $g(x)$ is evaluated. Comparing (24) with (18) and (21), the MUSIC-based imaging does not employ the physical model of the multistatic matrix except for the array manifold and the orthogonality between the signal and noise subspaces, thus could be less accurate if the model is accurate. Moreover, the MUSIC-based method cannot be extended to more complex models, such as unknown spatially correlated noise. The proposed statistically based estimator, instead, is more scalable in the sense that it could be extended to account for spatially correlated noise by employing a more realistic noise model as well as a more general physical model. All these extensions will finally result in more general optimization metrics, of course at the expense of higher computational complexity.

5. MULTIPLE-FREQUENCY LIKELIHOOD TIME-REVERSAL IMAGING

In this section, we generalize our fixed-frequency likelihood time-reversal imaging method to multiple frequencies by combining the imaging metrics at various frequencies. We demonstrate its usefulness in resolving the spatial ambiguity of a sparse array.

Our motivation for this generalization is to solve the dilemma in the tradeoff between the array size and resolution capability in the fixed-frequency methods by introducing a new degree of freedom. Using a homogeneous medium as an example, in order to keep the cross-range and range diffraction resolutions unchanged with the increasing of the range $R$, we need to increase the array aperture $a$ linearly accordingly, due to facts that the diffraction resolution of the refocused field in the cross-range direction is $\lambda R/a$, and $\lambda R/a^2$ in the range direction [3], [17]. This requires a large number of antennas when $R$ is large for arrays with antennas densely spaced. One possible solution to avoid the need for this large number of antennas is to employ a sparse array whose antenna spacing is much larger than half of the wavelength. However, it is well-known that a spatial aliasing will occur when we undersample in the spatial domain, and the introduced spatial ambiguity cannot be resolved without any a priori information.

Using multistatic matrices measured at multiple frequencies, we generalize the fixed-frequency likelihood time-reversal imaging methods to the multiple-frequency version by combining likelihood imaging metrics at different frequencies via multiplication, i.e., using the following metric for the multiple-frequency likelihood imaging

$$l_s(x; Y_1, Y_2 \cdots Y_L) = \prod_{l=1}^{L} l_s(x; \omega_l, Y_l),$$

$$= \prod_{l=1}^{L} \frac{1}{\|P_{l_t}^g(x; \omega_l)g_t(x; \omega_l)\|_F^2}$$

(25)

where $Y_l$ is the multistatic matrix measured at the $l$-th frequency $\omega_l, l = 1, 2, \ldots, L$, and $L$ is the number of total frequencies used. Intuitively, the imaging metric $l_s(x; Y_1, Y_2 \cdots Y_L)$ will peak at the position where $x$ coincides with the scatterer location, since every $l_s(x; \omega_l, Y_l)$ has a local maximum at the true scatterer location for $l = 1, 2, \ldots, L$; on the other hand, when $l_s(x; \omega_l, Y_l)$ is evaluated at the grating lobes for some frequency, there exists at least one frequency $\omega_{\ell'}$ such that $l_s(x; \omega_{\ell'}, Y_l)$ is small due to the different ambiguity patterns at different frequencies. More details are referred to our journal paper [8]. Observing that (25) is simply a product of the imaging metrics at every single frequency, the multiple-frequency image could be easily updated if more measurements are obtained at additional frequencies.

6. NUMERICAL EXAMPLES

For convenience of visualization, we consider the two-dimensional case, i.e., the antenna elements and scatterers are parallel lines and they are embedded in homogeneous background; then, from a mathematical point of view the locations of targets and antennas could be represented as points in $\mathbb{R}^2$. The Green function for this two-dimensional case is

$$G(r, x') = \frac{i}{4} H_0(2\pi|r - x'|/\lambda),$$

(26)

where $H_0$ is the zero order Hankel function of the first kind, see [18]. We will drop the unessential constant $i/4$ in the simulations. In each example, all the multistatic matrices are computed using
Fig. 2. Likelihood time-reversal image of point scatterers located at (-5,20), (0,25), and (5,20) using a ULA with wavelength $\lambda = 1$ and antenna spacing of half wavelength. *: antenna, ◦: scatterer.

(13), i.e., incorporating all the multiple scattering among the scatterers, and corrupted by the white Gaussian noise using (14). In all the numerical examples, we employ co-located ULAs as transmit and receive arrays and scatterers is assumed to have unit scattering coefficients.

In the first example, we demonstrate the time-reversal image using the likelihood imaging metric (23). The transmit and receive ULAs are located between (-20,1) and (20,1) with antenna spacing 1/2, the wavelength $\lambda = 1$, three scatterers, which are represented by "◦", locate at (-5,20), (0,25), and (5,20), respectively. The image is generated over the grid of $301 \times 301$ and the antennas are denoted as “*”. In Figure 2, the peaks indicate the correct scatterer locations. However, when the antenna spacing increases to 5 a spatial ambiguity appears in Figure 3. We can see a number of grating lobes in this image.

In the second example, seven frequencies are used in the multiple-frequency likelihood imaging, which correspond to the wavelength $\lambda = 0.60, 0.73, 0.87, 1, 1.13, 1.27, \text{ and } 1.4$. The simulation setup is the same as that of Figure 3, and the multiple-frequency likelihood image using the (25) is shown in Figure 4, from which we could see that the grating lobes are suppressed effectively and the true scatterer locations are resolved without ambiguities. Note that 81 antennas are used in Figure 2, whereas only 9 antennas are used in Figure 4. In the third example, we examine the performances of the ML and three sub-optimal estimators of the location parameters in terms of the mean-squared error (MSE). By 1500 Monte Carlo runs, we compute the MSEs of the MLE (15) which uses the full Foldy-Lax model, (21) in the likelihood imaging, (18) using the distorted wave Born approximation, and (19) that partially employs the Foldy-Lax model. The setups of the antennas, wavelength, and scatterers are the same as those in Figure 4. We assume knowledge of the scatterer locations in this example so that we do not have to resolve the spatial ambiguity of the sparse array when using only single frequency measurement. As expected, the likelihood imaging scheme has the largest MSE among the four estimators, and the MLE (15) based on full Foldy-Lax model performs the best. Note that the performance of the estimator based

Fig. 3. Likelihood time-reversal image of point scatterers located at (-5,20), (0,25), and (5,20) using a ULA with wavelength $\lambda = 1$ and antenna spacing of 5. *: antenna, ◦: scatterer.

Fig. 4. Multiple-frequency likelihood time-reversal image of point scatterers located at (-5,20), (0,25), and (5,20) using a ULA with wavelengths $\lambda = 0.60, 0.73, 0.87, 1, 1.13, 1.27, \text{ and } 1.4$ and antenna spacing of 5. *: antenna, ◦: scatterer.
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9. REFERENCES