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2.7 THE USE OF THE GAUSS–MARKOV THEOREM IN WINDS ANALYSIS*†

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1. INTRODUCTION

The FAA and NWS have deployed a number of atmospheric sensing systems near major airports. Traditionally, systems have been developed to use data from a single sensor. The FAA's Integrated Terminal Weather System (ITWS), currently in production, integrates the information from a number of weather sensors to provide more accurate and more consistent weather information for use at airports. This system will improve safety and efficiency of airport operations (Evans and Ducot, 1994). Another airport specific meteorological data fusion effort is underway for NASA's Aircraft Vortex Spacing System (AVOSS) (Dasey and Hinton, 1999).

Wind information is of particular importance in three areas: aircraft control, both by automation systems and by human controllers; storm evolution predictions; and adaptive spacing for wake vortex avoidance. There currently are a number of sources of wind information in the airport region: Doppler weather radars, surface anemometers, aircraft reports, and wind predictions from numerical weather models. It is the great abundance of Doppler data that drives the development of our winds analyses. Additional sensors are available at testbed sites such as the Dallas/Ft. Worth (DFW) airport: sonic wind profilers (SODAR), Doppler radar wind profilers, and an instrumented tower. The requirements for accuracy, timeliness, and information content for each use are different, so user specific algorithms must be developed.

A state–of–the–art analysis technique for producing gridded fields from non–Doppler meteorological data analysis is Optimal Interpolation (OI) (Gandin, 1963, Daley, 1991). Optimal Interpolation is a statistical interpolation technique that under certain hypotheses gives an unbiased minimum variance estimate. Differences between the observations and the initial estimate at the observation location are computed (Δ_j for the jth observation). The Δ_j terms are averaged in a least square sense to form a perturbation field which is then added back to the initial estimate. If the observations, as has traditionally been the case, are sparse relative to the desired resolution of the wind analysis this provides a method to adjust the overall wind field without smoothing over the structure in the initial estimate, which would occur if the sparse data were analyzed directly. This method ties the errors in the output field to the errors in the initial estimate, which is a reasonable trade–off when data are sparse. Standard OI implementations require observations of the full horizontal wind vector, which Doppler radars do not provide.

In traditional multi–Doppler wind analysis, radars are sited so that they cover the region of interest with significantly different viewing angles (Armijo, 1969). At a given location, each radar then provides an estimate of a different wind component. If two radars are used, a simple change of coordinates results in an estimate of the horizontal winds at that location in standard eastward (u) and northward (v) components. If three or more radars are used, the resulting system of equations is overdetermined, and the horizontal wind can be estimated using least squares techniques. When the geometry is good, and each radar has sufficient return power, the resulting wind estimates are very accurate. However, at locations without returns from at least two radars, this method can not be used. At locations where the radars are looking in nearly the same direction the solution to the equations is numerically unstable, and the method again can not be used. An operational system using existing radars cannot count on good Doppler returns where they are desired, nor can the system count on favorable radar siting.

We apply the Gauss–Markov Theorem (Luenburger, 1969) to develop a set of analyses to jointly analyze both vector quantities and single component quantities (i.e. Doppler measurements). The ITWS Terminal Winds (TW) gridded analysis provides a smooth transition between an analysis of differences from the initial estimate in data poor regions to a direct analysis of data in data rich regions. It is the ease with which the Gauss–Markov Theorem allows for such properties that motivated its use. This technique provides a new capability which is important since increasing numbers of Doppler weather radars are being deployed.

These analyses account for the differing errors in the wind information and the correlations among these errors. Highly correlated errors arise frequently due to the nonuniform distribution of data from the Doppler radars. If these correlated errors are not accounted for, these data dominate the analysis to a degree greater than is warranted by their information content.

2. THE GAUSS–MARKOV THEOREM

In order to apply the Gauss–Markov Theorem, we need to pose the problem in the form

(1) $A\boldsymbol{x} = \boldsymbol{d}$,

where $\boldsymbol{x}$ is the unknown wind vector, $\boldsymbol{d}$ is the data vector, and A is a linear transformation of the space of unknowns into the space of observations. In our applications, the components of the vector $\boldsymbol{x}$ are the eastward wind component u, the northward wind component v, and may also contain the spatial derivatives of these wind components.

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The vector $\boldsymbol{d}$ contains the wind information to be analyzed: observed wind components and information from previous analyses or from numerical weather prediction models.

The Gauss–Markov Theorem states that the linear minimum variance unbiased estimate of $\boldsymbol{x}$ is given by

(2) $\boldsymbol{x} = (A^T C^{-1} A)^{-1} A^T C^{-1} \boldsymbol{d},$

if each element of $\boldsymbol{d}$ is unbiased, and C is the error covariance matrix for the elements of $\boldsymbol{d}$. The error covariance of the solution is

(3) $(A^T C^{-1} A)^{-1}$.

While we are primarily interested in solving a vector problem, under certain conditions the wind vector problem reduces to two scalar problems, and some variables are intrinsically scalars. In the case that $\boldsymbol{x}$ is a scalar, the solution is especially simple. Let W be the matrix C^{-1}. The matrix A is a column of ones. Solving equation 2 we see that each value d_n gets a weight equal to the sum of the values w_{ij} in the n^{th} column divided by the sum of all values w_{ij}, and the value in equation 3 is just one over the sum of all w_{ij}.

3. TW INTERPOLATION TECHNIQUE

The technique used in the TW gridded analysis is applied independently at each grid point and has the following properties:

1. The analysis produces multi–Doppler quality winds in regions where multi–Doppler analyses are numerically stable.
2. The analysis is numerically stable in regions where multi–Doppler analyses are not numerically stable.
3. The analysis produces near multi–Doppler quality winds in small gaps in multi–Doppler radar coverage.
4. The analysis directly analyzes data in data rich regions and analyzes differences from the initial estimate in data sparse regions.
5. The analysis produces smooth transitions between regions with differing density of data.

Throughout this section we use the following notation:

- r denotes a radial wind component
- u denotes an east wind component
- v denotes a north wind component
- superscript i denotes a initial estimate quantity
- superscript o denotes an observed quantity
- subscripts denote location, o denoting an analysis location

In order to apply the Gauss–Markov Theorem, we need to pose the problem in the form of equation 1, where $\boldsymbol{x} = (u_o, v_o)^T$ is the unknown horizontal wind vector and $\boldsymbol{d}$ contains the initial wind estimate and information derived from observations in a window centered on the analysis location. The size of the data collection window adjusts dynamically based on local data density. The form of the matrix A depends on the number and type of data, vector and/or radial, to be analyzed.

When the data window contains vector observations and Doppler observations, in addition to the initial estimate, equation 1 has the form:

$$(4) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \cos\theta_n & \sin\theta_n \end{bmatrix} \begin{pmatrix} u_o \\ v_o \end{pmatrix} = \begin{bmatrix} u_o^i \\ v_o^i \\ u_m^o-(u_m^i-u_o^i) \\ v_m^o-(v_m^i-v_o^i) \\ r_n^o-(r_n^i-r_o^i) \end{bmatrix}$$

where rows 3 and 4 repeat for each observation of a horizontal wind vector and row 5 repeats for each Doppler observation.

The terms of the form $(f_m^i-f_o^i)$ are estimates of the displacement error in the variable f that arise from taking a measurement at location m and using that measurement as an estimate at location o. The displacement error is just the change in f between the location of the observation and the analysis point. The actual change is not known, so it is estimated from the initial estimate of the field f. The initial estimate of the radial component is computed from the initial estimates of u and v. The resulting estimates of the form $f_m^o-(f_m^i-f_o^i)$ are unbiased estimates of the variable f at the analysis location provided the observations are unbiased relative the observation locations. This is true even if the initial estimate has a bias, since differencing the initial estimate removes the bias.

In data rich regions, a small data window is employed, resulting in small displacement distances. This, coupled with the fact that the initial estimate is smoothed prior to applying the Gauss–Markov Theorem, causes the displacement error terms to be near zero in data rich regions; the observations in data rich regions are analyzed directly. This allows the analysis to incorporate the full richness of detail in the observations and de–couples the errors in the output field from the errors in the initial estimate. In data poor regions large data windows are used and the displacement terms come into full play. While the form of the analysis using the displacement error corrections is different from the form classical OI takes, it is equivalent: each is simply a different method of solving the same least squares problem, assuming a consistent set of error models.

Unlike the multiple Doppler analysis, the TW analysis is always numerically stable due to the inclusion of the initial estimate wind. The inclusion of a (u,v) data point provides two component estimates at right angles, giving a maximum spread of azimuth angles. Since the Doppler data are usually much more numerous than the other data, the TW solution closely matches the multiple Doppler solution at locations where the multiple Doppler problem is well conditioned. Otherwise, the analysis gives a solution that largely agrees with the radar observations in the component measured by the radars. The remaining component is derived from the vector estimates. A detailed comparison to the stability of dual doppler techniques is given in (Cole and Wilson,1994).

In practice, the error covariance matrix C is not known and must be estimated. There are two types of errors to estimate. The first is the error that arises from imperfect sensors or an imperfect initial estimate. The second is the error due to an imperfect correction of the displacement error. Our error models are based on the following simplifying assumptions:

1. Observations are unbiased.
2. Sensor errors from different observations are uncorrelated.
3. Errors in *u* and *v* components, observed or initial estimate, are uncorrelated.
4. Displacement errors and sensor errors are uncorrelated.
5. Displacement errors are functions solely of the horizontal, vertical, and temporal distance of the observation from the analysis point.

These assumptions are found to hold relatively well for our data sets. With these assumptions, the error covariance matrix *C* decomposes into the sum of a sensor error covariance matrix and a displacement error covariance matrix. The sensor error covariance matrix is diagonal, and the sensor error variances are well known. Note that if only wind vector observations are used, assumption 3 leads to a system that decomposes into two scalar systems, each with the same error covariance matrix.

The displacement error covariance model for two non–orthogonal, non–parallel components must take into account the angle between the two components. We denote the angle between the observed component and the *u* axis by θ, with east at 0^0, and north at 90^0, and the displacement error in observation *j* by δ_j. Then the displacement error covariance for two observations is given by the following equation:

(5) $\mathrm{Cov}(\delta_1,\delta_2) = \cos(\theta_1-\theta_2)[\mathrm{Var}(\delta_1)\mathrm{Var}(\delta_2)]^{1/2}\mathrm{Cor}(\delta_1,\delta_2)$

4. DOPPLER WIND PROFILES

The Gauss–Markov Theorem is also used to analyze Doppler data surrounding a radar to produce wind profiles. A Doppler Profile Analysis (DPA) is part of the real–time wind profiling system operational in DFW in support of a NASA wake vortex effort (Matthews and Denneno, 1999). In this system, data from two FAA Terminal Doppler Weather Radar (TDWR) are used to produce two wind profiles with the winds computed in headwind and tailwind components. These profiles have a vertical resolution of 50 m and an update rate of 5 minutes.

Each DPA uses data from only one TDWR. Data are collected around the radar at a fixed altitude from multiple elevation angles and the Gauss–Markov Theorem is applied. Since there is no gridded initial estimate from which to compute displacement errors, these correction terms are not used. The result is then an average wind over the data collection region as required by the AVOSS.

The numerical stability and the inherent quality estimates (equation 3) make this an attractive method for producing Doppler wind profiles. The NASA wake vortex system requires not only estimates of the horizontal winds, but also estimates of the expected variance of the winds from these estimates over (nominally) the next 30 minutes. Once the estimates are made it is straight forward to estimate the variance of the observations about each wind estimate. This variance is computed for a spatial window corresponding to a temporal average of 30 minutes. Since only Doppler data are used, this variance is a mix of the wind variability over all component directions; independent wind variability estimates for headwind and tailwind are not provided. The Gauss–Markov Theorem provides estimates of the error variance of each wind component. Since the error variances of the wind estimates and the wind variability are independent, their sum gives the desired estimates of the expected variances of the wind components about the wind estimates.

5. COMPUTATION OF WIND FIELD DERIVATIVES

A follow–on ITWS product is the prediction of convective storm evolution. This is an important product for improving traffic flow and planning. Divergence in the surface wind flow plays a key role in understanding storm evolution. Negative divergence, or convergence, indicates an updraft and increasing storm intensity. Conversely, a positive divergence in the surface flow arises from the storm core descending and a decrease in storm intensity. This gives rise to the need for a winds analysis that provides estimates of the spatial derivatives of the horizontal wind components.

The Initial Operating Capability (IOC) ITWS TW analysis was not tailored to modeling the surface flow; air traffic control is concerned with winds above the surface. The IOC TW analysis was tailored to minimize the error in the wind vectors, but was not tailored to minimize the errors in the derivative fields. A new version of the TW analysis is in development to estimate jointly the horizontal winds and the spatial derivatives of the horizontal winds, while minimizing the errors in the wind vectors and the errors in the derivatives.

Storm induced gust fronts, the outflow of cold air from thunder storms, are a primary source of surface convergence. While IOC TW does capture gust front signatures, the locations generally lag behind the actual locations due to temporal smoothing. The spatial smoothing causes the estimated magnitude of the convergence to be biased low. The IOC ITWS has a gust front detection algorithm that provides the location and velocity of gust fronts. This information will be used to project the initial estimate wind field forward in time to the analysis time, to appropriately adjust the error models used in the spatial smoothing (the displacement error models for horizontal displacements and temporal displacement), and to notify the analysis to use small data windows near the frontal boundary to avoid smoothing across the front.

A simple approach to estimating the spatial derivatives would be to take the difference of adjacent grid points and divide by the distance between them. This is problematic; if the same, or nearly the same, input data are used at the adjacent points their difference results not in an estimate of the change in wind between the two points, but a weighted difference of the error models used in the analysis. The resulting derivative estimate is then highly dependent on the error models. Averaging data to a grid point leads to a loss of precision in the location of the data. This in turn leads to poor derivative estimates since the change in wind values is divided by the distance between grid points, not the distance between observations.

A better approach is to estimate the derivatives directly from the data. The advantage is much less dependence on the error models used, the derivative estimates are always based on independent data values in an appropriately sized data window, and it removes the need for the displacement error correction terms used the IOC TW analysis. It also allows for a direct estimate of the errors in the resulting derivatives.

Now, when the data window contains vector observations and Doppler observations, in addition to initial estimates of the parameters being estimated, equation 1 has the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & \Delta x^o_m & 0 & \Delta y^o_m & 0 \\ 0 & 1 & 0 & \Delta x^o_m & 0 & \Delta y^o_m \\ c_n & s_n & \Delta x^o_n c_n & \Delta x^o_n s_n & \Delta y^o_n c_n & \Delta y^o_n s_n \end{pmatrix} \begin{pmatrix} u \\ v \\ u_x \\ v_x \\ u_y \\ v_y \end{pmatrix} = \begin{pmatrix} u^i \\ v^i \\ u^i_x \\ v^i_x \\ u^i_y \\ v^i_y \\ u^o_m \\ v^o_m \\ r^o_n \end{pmatrix} \quad (6)$$

where rows 7 and 8 repeat for each observation of a horizontal wind vector and row 9 repeats for each Doppler observation.

The unknowns are u, v, u_x, u_y, v_x, and v_y , where the subscripts denote derivatives. The terms Δx^o and Δy^o are the known distances between an observation and the analysis location, and c_n and s_n are the cosine and sine of the n^{th} radar observation azimuth angle. Terms of the form $f_x \Delta x + f_y \Delta y$ estimate the change in the field f between the observation location and the analysis location, and are analogous to the displacement error correction terms used in the IOC TW analysis. However, they are now estimated in the least–squares process using, in part, current observations. Thus they adjust more readily to changing conditions.

Rows 3–6 can be dropped from equation 6 if three or more initial estimate winds are included. The benefit of having the initial estimates of the derivatives is that this preserves peaks in their values, whereas substituting grid values tends to smooth peaks. The benefit of substituting grid values for the initial derivative values is that this eliminates complicating interactions of the models for wind errors and models for derivative errors. It appears that while it is theoretically appealing to use the initial estimates of the derivative, from a practical perspective, the use of grid wind values instead results in a better analysis.

6. COMPUTATION OF LARGE SCALE FEATURES

The TW system uses the Gauss–Markov Theorem to perform an analysis local to a grid point. In the current TW system, at a point with poor radar geometry and only Doppler data, the output wind estimate is largely an average of the local Doppler values in the component measured by the radars with the remaining component estimated from the large scale model used to initialize the analysis. The larger scale information in the Doppler data is not considered.

We are working on a extension to the TW system that uses the Gauss–Markov Theorem to compute wind estimates at a very large scale followed by successive refinements. At each step the data are divided into smaller domains and the previous estimates are used to initialize the next refinement. This allows for much better utilization of the Doppler data. We are currently working on an algorithm to incorporate the ITWS gust front detections so that the division of the data results in domains that do not cross gust front boundaries, thus retaining these features in the final analyses. The initial versions of this enhancement to the TW algorithm show great promise.

7. CONCLUDING REMARKS

The Gauss–Markov Theorem is a powerful tool for analyzing winds from multiple data sources. In particular, the Gauss–Markov Theorem handles Doppler data in an elegant fashion. Doppler data can be analyzed without transformation into problematic vectors. The Gauss–Markov Theorem takes into account varying quality of observations and the correlations in observation error. This later point is especially important when using Doppler data which is often nonuniformly distributed. The Gauss–Markov Theorem is also ideally suited to estimating wind field spatial derivatives.

We have built a number of real–time wind analysis systems based on the Gauss–Markov Theorem, one of which (ITWS Terminal Winds) is in production for the FAA.

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