The Performance of the Parametric Vector AR Adaptive Beamformer

Peter Parker
Michael Zatman

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Why We Care About PVAR

- PVAR converges faster than many other adaptive processing algorithms
  - but not as fast as the best Toeplitz covariance estimators
- Theoretical performance and robustness to some types of system errors were previously unknown
# Introduction

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✅ New PVAR results

Addressed here
Outline

• Signal Model and PVAR Algorithms
• Convergence of PVAR ABF
• Array Manifold Errors
• Limited Cancellation Ratio
**ABF Signal Model**

- $M$ signals incident on an $N$ element uniform linear array:

  \[ x(t) = As(t) + n(t) \]

  Noise

  Matrix of Steering vectors

  Signal modulations

  \[ A = [a(\theta_1), ..., a(\theta_M)] \]

  \[ a(\theta) = \frac{1}{\sqrt{N}} \left[ 1, e^{j2\pi \lambda d \sin(\theta)}, ..., e^{j(N-1)2\pi \lambda d \sin(\theta)} \right] \]

- Uncorrelated Gaussian signals in spatially white Gaussian noise

  \[ R = E\{x(t)x(t)^H\} = ASA^H + \sigma^2 I \]

  \[ S = E\{s(t)s(t)^H\} = \text{diag}[p_1, ..., p_M] \]

- $K$ samples of data available to estimate the covariance and train the weights

  \[ \hat{R} = \frac{1}{K} \sum_{k=1}^{K} x_k x_k^H \]
PVAR ABF

- PVAR uses ULA structure to improve convergence
- Signal model for ULA follows an $M^{th}$ order AutoRegressive model

$$\sum_{i=0}^{M} h_i x_{n-i} = \varepsilon_n \iff H^H x = e$$

Spatially white noise

$H$ is orthogonal to $x$

- PVAR projects signal onto subspace of $\hat{H}$

- Implementation

\[ \hat{R} = \frac{1}{K} \sum_{k=1}^{K} x_k x_k^H \]

Estimate spatial covariance

Spatial smoothing $L = M + 1$

\[ [\hat{U} \ \Lambda \ \hat{U}^H ] = \hat{R} \]

Eigen-decomposition

Eigenvector with minimum eigenvalue

Form projection

$$w_{PVAR} = P_{\hat{h}} v$$

Effective aperture is one more element than signals present

MIT Lincoln Laboratory

ASAP2002-6
PAP 5/10/02
STAP Notation

- $N$ element array
- $M$ pulses per coherent processing interval
- $MN \times 1$ space-time snapshot

$$\mathbf{X} = \begin{bmatrix}
  \mathbf{x}(1) \\
  \vdots \\
  \mathbf{x}(M)
\end{bmatrix}$$

- Space-time covariance

$$\begin{bmatrix}
  \mathbf{R}_0 & \cdots & \mathbf{R}_{M-1} \\
  \vdots & \ddots & \vdots \\
  \mathbf{R}^*_0 & \cdots & \mathbf{R}_{M-1}
\end{bmatrix}, \quad \mathbf{R}_i \text{ Spatial cross covariance } i \text{ pulses apart}$$
2-D PVAR STAP

- 2-D PVAR STAP
  - Uses uniform structure in space (ULA) and time (pulses)
  - 2-D autoregressive model

\[ \sum_{j=0}^{L_M} \sum_{i=0}^{L_N} h_{i,j} x_{n-i}(t - j) = \varepsilon_n(t) \]

Spatially and temporally white noise

\[ G^H X = e \]

- Implementation

Estimate space-time covariance → Spatial and temporal smoothing → Eigen-decomposition → Eigenvector with minimum eigenvalue → Form projection

\[ \hat{R} = \frac{1}{K} \sum_{k=1}^{K} X_k X_k^H \]

Size \( L_M L_N \times L_M L_N \)

\[ [\hat{U} \ \hat{U}^H] = \hat{R} \]

\[ \begin{bmatrix} \hat{h}_{1,i} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots \\ \hat{h}_{L_N,i} \end{bmatrix} \]

\[ \hat{H} = \begin{bmatrix} \hat{H}_1 \\ \vdots \\ \hat{H}_{L_M} \end{bmatrix} \]

\[ \hat{G} = \begin{bmatrix} \hat{G}_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots \\ \hat{G}_{L_M} \end{bmatrix} \]

\[ \mathbf{w}_{PVAR} = \mathbf{P}_G \mathbf{v} = \hat{G} \left( \hat{G}^H \hat{G} \right)^{-1} \hat{G}^H \mathbf{v} \]
1-D PVAR STAP

- 1-D PVAR STAP
  - Uses uniform structure in time (pulses)
  - 1-D vector autoregressive model

\[ \sum_{i=0}^{L} H_i x(t - i) = e(t) \quad \iff \quad G^H X = e \]

\[ N \times q \text{ matrix tap} \]
Not dependent on ULA

- Implementation

```
Estimate space-time covariance → Temporal smoothing → Eigen-decomposition → q smallest eigenvectors → Form projection
```

\[ \hat{R} = \frac{1}{K} \sum_{k=1}^{K} X_k X_k^H \quad \text{Size} \quad LN \times LN \]

\[ \begin{bmatrix} \hat{U} & \hat{\Lambda} & \hat{U}^H \end{bmatrix} = \hat{R} \]

\[ \begin{bmatrix} \hat{H}_1 \\ \vdots \\ \hat{H}_L \end{bmatrix} = U_{\text{min}} \]

\[ w_{PVAR} = P_{\hat{G}} v \]

\[ = \hat{G} \left( \hat{G}^H \hat{G} \right)^{-1} \hat{G}^H v \]

\[^1\text{STAR filter from previous ASAP}\]
A. L. Swindlehurst and P. Parker, ASAP2000
1-D PVAR and Pre-Doppler STAP

Optimized pre-Doppler\(^2\)

Temporal only filtering causes sub-optimal performance

- Estimate space-time covariance
  \[ \hat{R} = \frac{1}{k} \sum_{k=1}^{K} X_k X_k^H \]

- Select P pulse sub-CPI
- Perform temporal smoothing

- Estimate temporal clutter response

- Find temporal only orthogonal subspace (t)
  \[ w = \hat{R}_s^{-1} (t \otimes a) \]
  
  Sub-CPI STAP

- Sub-CPI recombination

1-D PVAR

- Estimate space-time clutter response

- Find orthogonal subspace

- Doppler filtering

\(^2\) E. Baranoski, ASAP 1996
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• Limited Cancellation Ratio
PVAR ABF and Root-MUSIC

Estimate spatial covariance → Spatial smoothing \( L = M + 1 \) → Eigen-decomposition → Eigenvector with minimum eigenvalue → Form projection

\[
\text{Root MUSIC} (\Theta) = \text{roots} \left( \sum_{i=0}^{M} u_i \gamma^i \right)^2
\]

\[
\gamma = e^{j2\pi \frac{d}{\lambda} \sin(\Theta)}
\]

- AR polynomial of PVAR is same as smoothed root-MUSIC
- PVAR places nulls in the direction of root-MUSIC estimates
- Use statistics of root-MUSIC and deterministic nulling to characterize PVAR

PVAR \( P_H \) = \( P_A(\Theta) \) Deterministic nulling in directions \( \Theta \)
Root-MUSIC Variance

- Closed form asymptotic expressions for variance of the root-MUSIC estimator\(^3\)
  - Assumes distinct eigenvalues for each signal present

\[
E\left\{ \left| \Delta \omega \right|^2 \right\} = S_{MU}^2 \frac{E\left\{ \left( \frac{d \hat{D}(e^{j\omega})}{d\omega} \right)^2 \right\}}{4N^2}
\]

\[
E\left\{ \left| \Delta z \right|^2 \right\} = S_{MU} \frac{E\left\{ \hat{D}(e^{j\omega}) \right\}}{N}
\]

\[
\left| \Delta r \right|^2 \approx \left| \Delta z \right|^2 - \left| \Delta \omega \right|^2
\]

For small errors

- Asymptotic distribution for DOA error (\(\Delta \omega\)) is Gaussian
- Asymptotic distribution for radial error (\(\Delta r\)) is \(\chi_1\)
  - Constrained to the unit circle for forwards/backwards averaging

SINR of Deterministic Nulling

\[
SINR = \frac{|w^H v|^2}{w^H R w} = \frac{P_t}{P_n + P_i}
\]

- \( P_t \) is the output target power
  \[
P_t = |v^H v - v^H DD^H v|^2
  \]
  Orthonormal basis of the estimated interference steering vectors

- \( P_n \) is output noise power
  \[
P_n = \sigma^2 (v^H v - v^H DD^H v)
  \]

- \( P_i \) is the unnulled jammer residue
  \[
P_i = v^H ASA^H v - 2 \text{Re}[v^H DD^H ASA^H v] + v^H DD^H ASA^H DD^H v
  \]

Components of SINR

30 dB Jammer at 0°
Deterministic null at 0.8°
Results*

10 Elements, 2 Jammers, 30 dB JNR, 5 Samples

Normalized SINR (dB)

SIN(Angle)

Results*

* Results shown are for a perfect array

Jammers at
sin(θ) = 0 and 0.7

- PVAR Theory
- PVAR MC
- Eig. Canc. Theory
Comparison

10 Elements, 2 Jammers, 30 dB JNR

- Toeplitz estimator provides better performance for closely spaced signals
- Smoothing increases the estimation error, especially for closely spaced sources
  - RMS error of MUSIC approaches the single signal CRB when the sources are about 1 (post-smoothing) beamwidth apart
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Array Manifold Error Model

- True steering vector is the ideal (assumed) steering vector with complex Gaussian errors
  \[ a_h(\theta) = a(\theta) \cdot h \]
  \[ h_i = 1 + \xi g_i \]
  Controls ‘size’ of the array manifold errors

- Mean sidelobe levels given by Ruze’s equation
  \[ S_{LL} = \frac{\xi^2}{N\eta} \]
  - approx. practical limit \( \xi^2 = -20 \text{ dB} (-10\text{dBi sidelobes}) \)

- Deterministic null depth with array manifold errors
  \[ \left| w^H a_h \right|^2 \approx 2 \left( 1 - \text{Re} \left[ a^H a_h \right] \right) \approx \frac{\xi^2}{N} \]

- Define mismatch as
  \[ 1 - \text{Re} \left[ a^H a_h \right] = \frac{\xi^2}{2} \]
  Deterministic Null Depth \( \approx \frac{2 \text{ Mismatch}}{N} \)
ABF Performance with Array Errors

- Toeplitz covariance estimation and PVAR perform poorly in the presence of array manifold errors and strong interference

- “Knee” in curve occurs when JNR = Null Depth (at –33 dB mismatch)
STAP Performance with Array Errors

Array mismatch of –26 dB (–46 dB deterministic null depth)
CNR = 33 dB per element/pulse (56 dB total)

12 Elements, 18 Pulses, 3 X DOF Samples

2-D PVAR STAP breaks down with array mismatch
1-D PVAR STAP is fully adaptive in space – maintains performance
Outline

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Limited CR

- Interference mitigation is limited by how well the channels are matched

- 2 channel cancellation ratio

\[ CR \triangleq 1 - \frac{\left| x_1^H x_2 \right|^2}{\|x_1\|^2 \|x_2\|^2} \]

- Adaptive null depth is limited to CR
  - Performance is good when CNR < Cancellation Ratio
STAP Performance with Limited CR

For limited CR
- Clutter discreet breaks through in 1-D PVAR and eigencanceller
- Pre-Doppler is able to null sidelobe clutter

For CR limited systems: Cancellation is better when some clutter is filtered prior to STAP
Summary

- PVAR beamformer is equivalent to smoothed root-MUSIC with deterministic nulling
  - Theoretical convergence derived for PVAR ABF
  - Good Toeplitz covariance estimators outperform PVAR ABF (by using entire aperture to estimate signal locations)

- 1-D PVAR STAP is suitable for airborne radar systems not limited by the spatial cancellation ratio
  - Assumes that pulse timing errors are very small. If the pulse timing errors are large then the algorithm will break down.

- Optimized pre-Doppler STAP best for cancellation ratio limited systems
  - Temporal only suppression of the clutter prior to the space-time adaptive processing
Backup Viewgraphs
Expected SINR

- SINR expression does not simplify very well (even for a single source)
  - Evaluate integral numerically

\[
\int_{x=-\infty}^{\infty} \int_{y=0}^{\infty} \text{SINR}(x, y) f_{\Delta \omega_i}(x) f_{\Delta r_i}(y) dx dy
\]

- \( f_{\Delta \omega_i}(x) \) **Gaussian distribution**
  - mean = 0
  - \( \text{var} = E\{|\Delta \omega_i|^2\} \)

- \( f_{\Delta r_i}(y) \) **\( \chi \) distribution** (forwards averaging only)
  - \( \text{var} = E\{|\Delta r_i|^2\} \)
  - DOF = 1

F/B averaging constrains the signal to the unit circle (i.e. var of radial error is zero)
Monte Carlo vs. Theory

- Why the Monte Carlo simulation has better performance than theory suggests for closely spaced signals

- $\sin(\theta) = 0$ and $-0.1$

- PVAR Theory
- Eig. Canc. Theory
- SMI Theory
- Unsmoothed MUSIC

- Histogram Gaussian
- High tail away from other signal

- This mass pushed toward center of distribution
Root-MUSIC Distribution

- Central limit theorem
  - Distribution of DOA error is asymptotically Gaussian
  - Distribution of radial error is asymptotically $\chi_1$ (for maximal smoothing)
- Histogram of DOA and radial errors for a single source
  - JNR=30 dB, N=10 elements
Pre-Doppler and Limited CR

- Temporal nulling can reduce strong sidelobe clutter below the CR limit prior to STAP
CR Limited Post-Doppler

- PRI-staggered post-Doppler and loaded SMI exhibit same behavior as other STAP algorithms when limited by CR

Strong clutter discrete located in training data (−0.37 Doppler)

12 Elements, 18 Pulses, 3 X DOF Samples, Array mismatch = -26 dB

1-D PVAR
PRI-staggered
Pre-Doppler
Loaded SMI